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# A BOUNDARY-ELEMENT APPROACH TO THE SOLUTION OF THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY BY THE GEOMETRICAL IMMERSION METHOD†

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A basic formulation of the geometrical immersion method (GIM) for solving three-dimensional boundary-value problems in the differentially formulated theory of elasticity is given. If the canonical domain is taken to be the entire Euclidean space, the differential formulation reduces to the corresponding boundary integral equation whose kernel is the Kelvin–Somigliana tensor. The integral equation obtained is realized numerically using the boundary-element approximation. Numerical experiments confirm the theoretical convergence of the GIM iterative process. The efficiency of this approach compared with the traditional methods of boundary integral equations for solving three-dimensional problems on the theory of elasticity is due to the absence of computationally intensive steps which invert densely-packed matrices of the influence coefficients in the direct solution of algebraic systems of equations, and the choice of parameters that ensure convergence when iterative methods are used.

Together with the finite-difference and finite-element methods, the boundary-element method (BEM) is widely used to calculate the stress–strain state of three-dimensional structures. It has a number of advantages [1], and, in particular, reduces the dimension of the original problem and, in consequence, reduces the order of the system of linear algebraic equations (SLAE) to be solved. However, in the BEM method the most laborious stage remains the solution of the SLAE with a densely-packed non-symmetric matrix of influence coefficients.

Below we describe a boundary-element implementation of the differential formulation of the geometrical immersion method (GIM),‡ which, on the one hand, enables one to preserve all the positive aspects of the BEM, and on the other hand replaces the process of solving the SLAE directly by an iterative procedure with guaranteed convergence, which leads to a significant saving in computing resources.

## 1. DIFFERENTIAL FORMULATION OF THE GEOMETRICAL IMMERSION METHOD

Consider an elastic isotropic body occupying the domain $D$ in Euclidean space $R^3$ with boundaries

$$S=\bigcup_1^N S_i$$

(Fig. 1). It is required to find the displacement vector $u(x)$ of the theory of elasticity boundary-value problem

$$\operatorname{div}\boldsymbol{\sigma}(\mathbf{u})+\mathbf{f}=0,\quad \mathbf{x}\in D;\quad \mathbf{n}(\mathbf{x})\cdot\boldsymbol{\sigma}(\mathbf{x})=\mathbf{g}(\mathbf{x}),\quad \mathbf{x}\in S \tag{1.1}$$

where $\mathbf{u}$, $\mathbf{f}$ and $\mathbf{g}$ are, respectively, the displacement vector, and the body and surface forces, $\mathbf{x}$ is the position vector of any point in the domain $D$, $\mathbf{n}$ is the outward normal to the boundary $S$, and $\sigma$ is the stress tensor.

Hooke's law and Cauchy's relations have the form

$$\boldsymbol{\sigma}=\lambda\theta\mathbf{E}+2\mu\mathbf{e},\quad \mathbf{e}=[(\nabla\mathbf{u})^T+\nabla\mathbf{u}]/2 \tag{1.2}$$

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‡SHARDAKOV, I. N. The geometrical immersion method for solving three-dimensional problems of the theory of elasticity. Doctoral dissertation, Moscow, 1990.

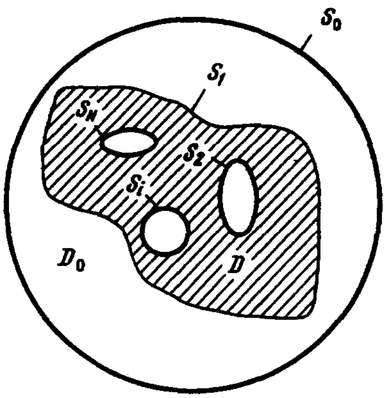


Fig. 1.

where $E$ is the unit tensor, $\theta$ is the first invariant of the deformation tensor $\mathbf{e}$, and $\lambda$ and $\mu$ are the Lamé parameters.

A generalized solution of boundary-value problem (1.1) may be obtained from a variational equation of the form

$$\int_D \sigma(\mathbf{u})\cdot\cdot\,\mathbf{e}(\mathbf{v})dD = \int_D \mathbf{f}\cdot\mathbf{v}dD + \int_S \mathbf{g}\cdot\mathbf{v}dS \quad \forall \in \mathbf{v}V(D) \tag{1.3}$$

which gives a minimum to the quadratic functional for the general potential energy of the elastic body

$$\Phi(\mathbf{v}) = \int_D \sigma(\mathbf{u})\cdot\cdot\,\mathbf{e}(\mathbf{v})dD - 2\int_D \mathbf{f}\cdot\mathbf{v}dD - 2\int_S \mathbf{g}\cdot\mathbf{v}dS \tag{1.4}$$

Here $V(D) \subset (H^1(D))^n$ is a complete closed subspace of the Sobolev vector functions $(H^1(D))^n$ [3].

The GIM asserts the possibility of establishing a relation between the solution of Eq. (1.3) and another complete closed subspace of vector functions

$$V_0(D_0) = \{\mathbf{u} \in (H^1(D_0))^n | \mathbf{u} = 0, \quad \mathbf{x} \in S_0^1 = S_0 \cap S_\Delta\}$$

each element of which is defined in the domain $D_0$, and also enables one to write a variational equation for the vector $w$

$$\int_{D_0} \sigma(\mathbf{w})\cdot\cdot\,\mathbf{e}(\mathbf{v})dD_0 = \int_{D_\Delta} \sigma(\mathbf{w})\cdot\cdot\,\mathbf{e}(\mathbf{v})dD_\Delta + \int_D \mathbf{f}\cdot v dD + \int_S \mathbf{g}\cdot\mathbf{v}dS \tag{1.5}$$

$$\forall \mathbf{v} \in V_0(D_0)$$

Here $D \subset D_0$, i.e. the original domain $D$ is completely contained within the domain $D_0$ (which accounts for the name of the method), $S_0$ is the boundary of $D_0$, $D_\Delta = D_0/D$ is the complement of $D$ in $D_0$ (see Fig. 1), $S_\Delta$ is the boundary of the domain $D_\Delta$, and $S_0^1$ is that part of the boundary $S_\Delta$ which belongs to $S_0$.

It has been shown (see the second footnote on p. 235) that when $x \in D$ the solution $\mathbf{w}$ of Eq. (1.5) gives the minimum of the functional (1.4) and is therefore a generalized solution of the original Eq. (1.3), i.e. $\mathbf{w} = \mathbf{u}$ when $\mathbf{x} \in D$.

We will solve the variational equation (1.5) by an iterative method, replacing $\mathbf{w}$ with $\mathbf{w}^k$ on the left-hand side and $\mathbf{w}^{k-1}$ on the right, with $k = 1, 2, \ldots$; $\mathbf{w}^0 = 0$. Applying the Gauss–Ostrogradskii theorem, we obtain the corresponding differential formulation of the problem

$$\operatorname{div}\sigma(\mathbf{w}^k) = -H(D)f - \Gamma(S_1)[\mathbf{g} + \mathbf{n}^\Delta\cdot\sigma^\Delta(\mathbf{w}^{k-1})], \quad \mathbf{x} \in D_0 \tag{1.6}$$

$$\mathbf{n}\cdot\sigma(w^k) = \mathbf{g}; \quad \mathbf{x} \in S \cap S_0; \quad \mathbf{w}^k = 0, \quad \mathbf{x} \in S_0^1$$

where $H(D)$ is a generalized Heaviside-type function [4] equal to 1 when $\mathbf{x} \in D$ and 0 when $\mathbf{x} \in D_\Delta$, $\Gamma(S_1)$ is a generalized Dirac-type function [4] centred on the boundary $S_1 = S \cap S_\Delta$, $\mathbf{n}^\Delta$ is the outward unit normal to $S_1$ relative to $D_\Delta$, and $\sigma^\Delta$ is the stress tensor in the limit as $\mathbf{x}$ approaches $S_1$ from the complement $D_\Delta$.

It has been shown (see the second footnote on p. 235) that the sequence $\{\mathbf{w}^k\}$ converges everywhere in the norm of $V_0(D_0)$, and therefore in the norm of $V(D)$, irrespective of how different the original domain $D$ is from the domain $D_0$ in which it is "immersed".

If the entire Euclidean space $R^3$ is chosen to be the canonical domain $D_0$, then an associated integral equation may be written for Eq. (1.6), in which Green's function $\mathbf{G}(\mathbf{x}, \boldsymbol{\xi})$ (the Kelvin–Somigliana tensor [2]) serves as the inverse operator to the boundary-value problem

$$\mathbf{w}^k(\mathbf{x}) = \int_D \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) dD(\boldsymbol{\xi}) + \int_S \mathbf{G}(\mathbf{x}, \boldsymbol{\xi}) \cdot [\mathbf{g}(\boldsymbol{\xi}) + \mathbf{n}^\Delta \cdot \boldsymbol{\sigma}^\Delta (\mathbf{w}(\xi)^{k-1})] dS \quad (1.7)$$

Here $\boldsymbol{\xi}$ is the point of the domain at which integration is performed.

Using standard methods of potential theory [1, 2], physical relations and Cauchy's relations (1.2), one can change from (1.7) to the boundary integral equations for the forces at the boundary of the domain under consideration

$$\mathbf{t}^k(\mathbf{x}) = \frac{1}{2}[\mathbf{g}(\mathbf{x}) + t(\mathbf{x})^{k-1}] + \int_S \mathbf{F}(\mathbf{x}, \boldsymbol{\xi}) \cdot [\mathbf{g}(\mathbf{x}) + \mathbf{t}(\mathbf{x})^{k-1}] dS + \int_D \mathbf{F}(\mathbf{x}, \boldsymbol{\psi}) \cdot f(\boldsymbol{\psi}) dD(\boldsymbol{\psi}) \quad (1.8)$$

Here $\mathbf{x}$ and $\xi$ are points on the boundary $S$, $\psi$ is a point in the interior of $D$, $\mathbf{t}^k(\mathbf{x})$ is the unknown force vector on the surface of the body at the $k$th iteration acting from the side of the complement $D_\Delta$, $\mathbf{t}^0(\mathbf{x}) = 0$, and $\mathbf{F}(\mathbf{x}, \boldsymbol{\xi})$ is a singular kernel which is a second-rank tensor.

Relation (1.8) enables us to determine directly the vector of the unknown forces at the boundary of $S$ for a class of problems with specified conditions on the stresses. Using boundary integral equation methods one can obtain equations for the displacements, strains and stresses both inside the domain and on the boundary.

The iterative integral equation (1.8) is the fundamental equation of the boundary-element implementation of the differential formulation of the GIM. To obtain a discrete analogue to Eq. (1.8) we apply the extensively-developed procedures of the BEM, which removes the need to solve the SLAE for the unknown variables. The solution $\mathbf{t}^k$ is determined by simply substituting the vector $\mathbf{t}^{k-1}$ found at the preceding iteration into the right-hand side of (1.8) ($k = 1, 2, 3, \ldots, L$), where $L$ is the number of completed iterations.

## 2. NUMERICAL IMPLEMENTATION

A finite-dimensional analogue of Eq. (1.8) is constructed in accordance with the BEM approach [2]. The boundary $S$ of the domain $D$ is approximated by a set of $N$ boundary elements. The boundary elements are discontinuous eight-cornered elements with quadratic coordinate approximation and with the unknown function represented in the form of a Lagrange polynomial of the first degree. It was pointed out in [1] that an additional advantage of this discretization is the possibility of easily combining elements of different shapes, because the approximations to the unknown functions do not have to be consistent between the elements. To compute the volume integral in (1.8) the interior of $D$ was decomposed into $M$ hexahedral cells with quadratic coordinate approximation and an integrable volume force function $f$.

The integration of the singular functions with a strong singularity at the surface of the element requires special techniques and integration schemes and is performed numerically with a non-regular distribution of a set of Gaussian points along the curvilinear surface of the three-dimensional body. The volume integral in (1.8) is the integral of a function with a weak singularity [2] and is therefore computed in the usual manner using Gaussian quadrature.

We write the discrete iterative matrix equation corresponding to (1.8) in the following form

$$\{t\}^k = \frac{1}{2}\left(\{t\}^{k-1} + \{g\}\right) + [F^{ss}](\{t\}^{k-1} + \{g\}) + [F^{s\upsilon}]\{f\} \quad (2.1)$$

The superscripts $s$ and $\upsilon$ denote quantities that are obtained from data relating only to surface points ($ss$) or to points on the surface and in the volume ($s\upsilon$). At the first step of the solution of the iterative equation the matrix coefficients of $\mathbf{F}^{ss}$, $\mathbf{F}^{s\upsilon}$ and the derivatives of the matrix $\mathbf{F}^{s\upsilon}$ along the vector $\mathbf{f}$ are computed. $L$ iterations are then performed with the zeroth approximation $\mathbf{t}^0 \equiv 0$. The convergence of the iterative process is followed using the mean-square difference of two consecutive approximations: at the $k$th iteration

$$\delta^k = \|\mathbf{t}^k - \mathbf{t}^{k-1}\| / \|\mathbf{t}\|^k \quad (2.2)$$

where $\|\cdot\|$ is the mean-square norm.

## 3. CONVERGENCE OF THE GEOMETRICAL IMMERSION METHOD

It has been shown (see the second footnote on p. 235) theoretically that the iterative procedure of the GIM converges everywhere, irrespective of how different the original domain $D$ is from $D_0$. Numerical experience shows that the rate of convergence of the solution to the discrete iterative equation (2.1) depends to a large degree on factors like the precision of the integration of singular integrals over the surface, the level of discretization of the surface of the body, and the ratio of the volume of the body to its surface. On the whole one can say that a sufficient level of solution accuracy is achieved by integrating the surface integrals using a scheme containing 175 Gaussian points for singular (including the singular point) boundary elements, 100 points for elements close to the singular point, and 25 integration points for elements far from the singular point. To obtain a satisfactory solution inside the domain the typical size of the boundary elements should not exceed 10% of the typical size of the object.

Figure 2 shows convergence curves for the iterative GIM process for different levels of domain boundary discretization. The convergence parameter was chosen to be the relative variation of the quantity $\delta$ (2.2) $\varepsilon^k = |(\delta^k - \delta^{k-1})/\delta^k|$, where $k$ is the number of the iteration. The numerical experiment being performed was for the problem of a long cylinder subjected to unit external pressure along the lateral surface. (The cylinder had radius 1, length 48, shear modulus 1 and Poisson's ratio 0.3.) The surface of the cylinder was decomposed into 12, 15, 20 and 24 boundary elements (curves 1–4, respectively).

As can be seen from curves 1–3, at a certain iteration the iterative GIM process diverges and it is senseless to continue. In this case the GIM also gives an erroneous solution which differs from the Lamé solution for an infinitely long cylinder by more than 100%. To explain this phenomenon it is necessary completely to suspend the procedure for solving the SLAE and computing the solution at points of the domain. With a sufficiently good decomposition of the cylinder surface (curves 3 and 4) the convergence of the GIM iterations is good, and the resulting solution after 40 iterations differs from the solution of the BEM to the limits of the accuracy of the BEM procedure, and the error relative to the Lamé solution does not exceed 2%. Hence the nature of the convergence of the iterative GIM procedure characterizes the discretization quality of the boundary of the object under consideration.

Figure 3 shows the convergence of the solution (the displacement $u_x$ along the $X$ axis) to the problem of a cube subjected to unit external pressure over its entire surface, according to the level of the boundary discretization. (The origin of coordinates is at the centre of the cube, the length of a side is 2, the shear modulus is 1 and Poisson's ratio is 0.3.) The curves shown are the relative error $\varepsilon = |(u_x - u_x^0)/u_x^0| \times 100\%$ along the $x$ axis (where $u_x^0$ is the exact solution): curves 1–3 correspond to 6, 24 and 54 elements.

## 4. SOLVING PROBLEMS. TEST PROBLEMS

To check the reliability of the results obtained using the GIM, a range of test problems was solved, and comparisons were made with existing analytic solutions. Figure 4 shows the solution to the problem of a hollow cylinder loaded with unit pressure along the external lateral surface. The displacements $u_r$ and stresses $\sigma_r$, $\sigma_\varphi$ were computed over the middle transverse section along the radius $r$ (the dashed curves). The calculated values were compared with the Lamé solution for a hollow cylinder in a plane stressed state (the corresponding continuous curves). The external radius of the cylinder was 6, the internal radius was 1, the length of the cylinder was 5, the shear modulus was 1, and Poisson's ratio was 0.3.

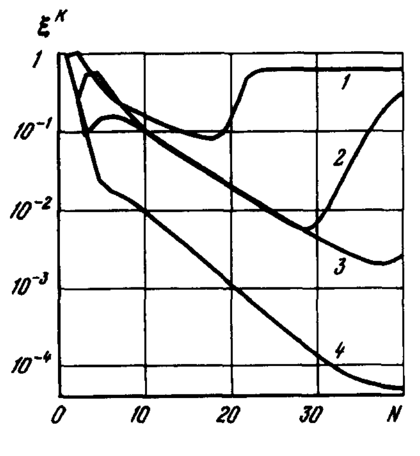


Fig. 2.

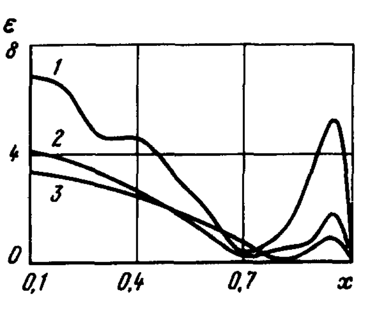


Fig. 3.

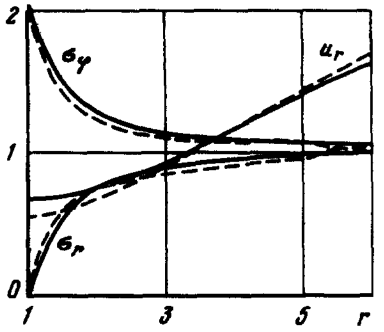


Fig. 4.

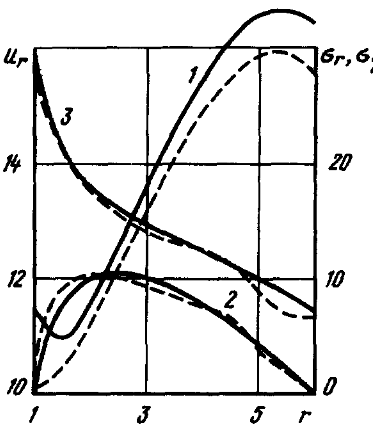


Fig. 5.

The solution of the problem for a rotating disk with the same geometrical and physical parameters as in the preceding problem (the angular velocity was 1 and the density was 1) is shown in Fig. 5. The displacements $u_r$ and stresses $\sigma_r$, $\sigma_\varphi$ were computed over the median cross-section along the radius $r$ (the dashed curves 1–3, respectively). The calculated values were compared with the analytic solution obtained in [5] (the corresponding continuous curves). It is clear from the results shown in Fig. 5 that the agreement between the calculated and analytically obtained solutions is very good.

*Solution of the problem for a perforated cylinder.* To illustrate the effectiveness of the boundary-element implementation of the differential formulation of the GIM when calculating the stress–strain state of three-dimensional structures we solved the problem for a cylinder with an internal star-shaped channel. Figure 6 shows the numerical scheme for the domain under consideration. An eighth of the object was taken to be the computational domain. The surface was decomposed into 14 boundary elements, and in the volume integral the interior domain was decomposed into 64 three-dimensional cells. The solution for the median $z = 0$ cross-section of the cylinder was investigated along the radius $r$ in two sections with angular coordinates $\varphi = 0°$ and $\varphi = 45°$.

Figure 7 shows the results of the solution of the problem with specified unit external pressure over the lateral surface of the cylinder (modulus of elasticity 100, Poisson's ratio 0.3): the displacements $u_r$ and stresses $\sigma_r$ and $\sigma_\varphi$ with $\varphi = 0°$ (curves 1–3) and along $\varphi = 45°$ (curves 4–6, respectively).

Figure 8 shows the results obtained for a problem with the same cylinder rotating at constant angular velocity (angular velocity 10, density 1): the displacements $u_r$ and stresses $\sigma_r$ and $\sigma_\varphi$ along $\varphi = 0°$ (curves 1–3) and along $\varphi = 45°$ (curves 4–6, respectively).

## 5. CONCLUSIONS

It has been shown that the main advantage of our proposed GIM method compared to the BEM method is that it removes the need to perform a laborious inversion operation on the non-symmetric densely-packed solution matrix. The GIM replaces this operation with a highly productive iterative process which does not require the selection of any parameters to ensure convergence (which is the case in methods traditionally used to solve SLAEs for every specific case). At the same time the GIM retains all the well-known advantages of the BEM.

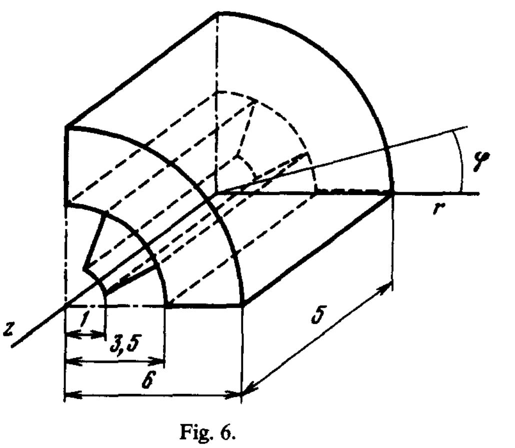


Fig. 6.

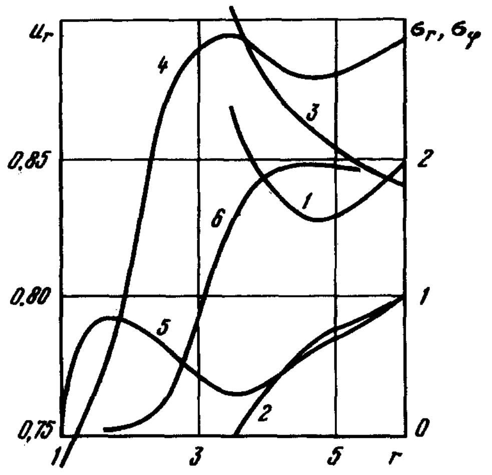

Fig. 7.

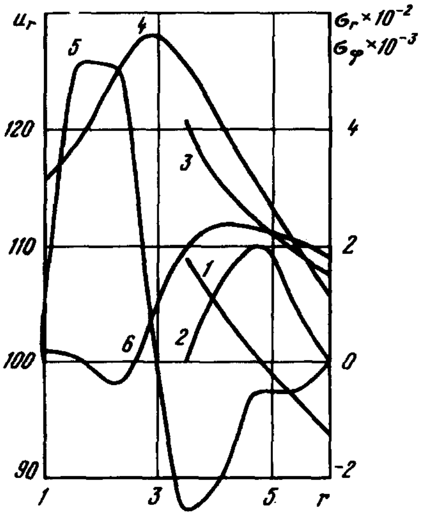

Fig. 8.

Table 1

| Number of elements | Number of unknowns | Time to set up the SLAE, s | Time to solve the SLAE, s | |
|---|---|---|---|---|
| | | | BEM | GIM |
| 6 | 72 | 14 | 4.7 | 3.9 |
| 24 | 288 | 93 | 267 | 45 |
| 54 | 648 | 320 | 2860 | 217 |

Table 1 shows the dependence of the processor time for the boundary element method and the geometrical immersion element on the number of unknowns in the SLAE. It should be noted that even with identical expenditure on setting-up the solution matrix and calculations for the internal points of the domain, the time taken to obtain the vector of unknowns in the GIM is significantly less than in the BEM. The saving in computer time is greater, the greater the dimension of the system. The results of the numerical experiment are given for an IBM PC AT/386 20 MHz.

The iterative solution of equations in the GIM enables one to use a procedure for accelerating the convergence of the process, it is easy to mesh the RAM to the partitioned form of the matrix of the influence coefficients at rigid supports, and to apply parallel computation algorithms.

Carefully based on theoretical considerations and developed in algorithms and programs, the GIM enables one to solve a wide class of three-dimensional problems with complex geometrical configurations, and its high efficiency and economy has been confirmed.

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